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Non-commutativity of space-time and the hydrogen atom spectrum

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Abstract. There has been disagreement in the literature on whether the hydrogen atom spectrum receives any tree-level correction due to non-commutativity. Here we shall clarify this issue and show that indeed a general argument on the structure of the proton as a non-elementary particle leads to the appearance of such corrections. As a showcase, we evaluate the corrections in a simple non-relativistic quark model with a result in full agreement with the previous one we had obtained by considering the electron moving in the external electric field of proton. Thus the previously obtained bound on the non-commutativity parameter, $\theta < (10^4\,\mathrm{GeV})^{-2}$, using the Lamb shift data, remains valid.

Recently a large amount of research work has been devoted to the study of physics on non-commutative space-times and in particular the non-commutative Moyal plane (for a review see, e.g., [1]). In these works both quantum mechanics (QM) and field theory on non-commutative spaces have been studied. Besides the theoretical interest, by comparing the results of the non-commutative version of the usual physical models with the present data, lower bounds on the non-commutative scale Λ_{NC} have been obtained [2–4]: as a conservative estimate, we have $\Lambda_{\mathrm{NC}} \gtrsim 1$–$10\,\mathrm{TeV}$.

In this paper we would like to focus on the hydrogen atom in the non-commutative space and re-analyze its spectrum. This system has previously been considered in [5, 6] with disagreement on the results. Here, through a more careful analysis we intend to clarify the discrepancy. In [5] we have analyzed the hydrogen atom in the non-commutative space considering the system described by a one-particle Schrödinger equation. Explicitly we considered the electron in an external Coulomb field; hence the system is described by

$$H|\psi\rangle = \mathrm{i}\hbar \frac{\partial}{\partial t}|\psi\rangle$$

with

$$H = \frac{\hat{p}_e^2}{2m_e} + V(\hat{x}_e)\ , \tag{1}$$

where $\hat{x}_e$ and $\hat{p}_e$ are the phase space coordinates of the electron and

$$\left[\hat{x}_e^i, \hat{x}_e^j\right] = \mathrm{i}\theta^{ij}\ , \tag{2}$$

$$\left[\hat{x}_e^i, \hat{p}_e^j\right] = \mathrm{i}\hbar\delta^{ij}\ , \tag{3}$$

$$\left[\hat{p}_e^i, \hat{p}_e^j\right] = 0\ . \tag{4}$$

Then it is easy to see that the new coordinates $x_i = \hat{x}_i + \frac{1}{2\hbar}\theta_{ij}\hat{p}_j$, $p_i = \hat{p}_i$ satisfy the usual canonical commutation relations [5]:

$$\left[\hat{x}_e^i, \hat{x}_e^j\right] = 0\ , \tag{5}$$

$$\left[\hat{x}_e^i, \hat{p}_e^j\right] = \mathrm{i}\hbar\delta^{ij}\ , \tag{6}$$

$$\left[\hat{p}_e^i, \hat{p}_e^j\right] = 0\ , \tag{7}$$

and in terms of these "canonical" coordinates, the Hamiltonian takes the familiar form of the usual hydrogen atom plus non-commutative corrections (cf. (2.5) in [5]),

$$V(r,p) = -\frac{Ze^2}{r} - Ze^2\frac{\mathbf{L}\cdot\theta}{4\hbar r^3} + O(\theta^2)\ , \tag{8}$$

where $\theta_i = \epsilon_{ijk}\theta_{jk}$, $r = \sqrt{\sum_i x_i^2}$ and $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. (The value of $|\theta|$ is the inverse square of the non-commutative scale Λ_{NC}.) From this we concluded that there exist non-commutative corrections to the spectrum and comparing our results with the data for the Lamb-shift experiments, we obtained the bound $\Lambda_{\mathrm{NC}} \gtrsim 10\,\mathrm{TeV}$.

On the other hand, in a more detailed analysis, the nucleus (here the proton) which exerts the Coulomb potential should also be considered as a dynamical object. In other words, one should solve the two-body Schrödinger equation. For the non-commutative hydrogen atom this has been done in [6]. There it was assumed that the proton, similar to the electron, is described by NC QED [7,8]. Based on this assumption and the fact that under charge conjugation θ_{ij} changes sign [9], it was shown that the non-commutativity effects will not change the spectrum of this two-body problem at the tree level (cf. (28) of [6]), in disagreement with the results of [5] (in particular, with (8)).

In this note we argue that in fact the very basic assumption of [6] that the effective non-commutativity parameter for the proton is equal to that of the electron with a minus sign is not physically valid. Therefore, the "cancellation" of non-commutativity effects is not complete and hence our previous results on the form (8) for the potential with the correction term, as well as on the lower bound on $\Lambda_{\rm NC}$ are indeed valid. The essential point is that the proton, due to the fact that it has structure and is a composite particle, cannot be described by NC QED (applicable to elementary particles). To systems such as positronium, however, the analysis of [6] is applicable, resulting in no corrections to the spectrum at the tree level, due to the non-commutativity of space-time. Noting the conservative bounds on $\Lambda_{\rm NC} \gtrsim 1$–$10\,{\rm TeV}$ obtained from another physical analysis [2,3], and that $\Lambda_{\rm QCD}$, or the inverse of the proton size, is of the order of 200 MeV, we notice that $\frac{\Lambda_{\rm QCD}}{\Lambda_{\rm NC}} \ll 1$. In other words, the QCD effects (here the internal structure of the proton) become important much before the non-commutative effects. In short, the proton in the non-commutative hydrogen atom essentially behaves as a *commutative* particle.

A full analysis of the problem, however, needs a better understanding of the non-commutative standard model (NCSM). Unfortunately, despite several efforts in constructing such a model [10,11], a complete formulation of NCSM is not yet available. Therefore, to present a quantitave treatment of the issue with the above arguments, we try to tackle the problem of finding an effective description of electromagnetic interactions of the proton through a naive quark model. In such a model we can safely assume that inside the proton we deal with free quarks.

However, there still remains a major problem to be addressed: in the NCQED the only possible charges coupled to the photon are $\pm 1, 0$ [7,12]. As a result, quarks (with fractional charges) cannot be described by NCQED. Nevertheless, since we are interested only in first order effects in θ, we can use NCQED vertices for quarks,[1] though only up to first order in θ. Then the effective electron–proton interaction is the sum of the electron–quark Coulomb potentials for the u, u and d quarks, namely,

$$V = V_{u_1} + V_{u_2} + V_d \ . \tag{9}$$

The expressions for the potentials V_q can be obtained, as usual, as the non-relativistic limit of non-commutative one-photon exchange,

$$V_q = -Qe^2 V\left(\mathbf{x}_q + \frac{1}{4}\theta \times \mathbf{K}_q\right) \ , \tag{10}$$

where $V(\mathbf{r}) = \frac{1}{|\mathbf{r}|}$, Q is 2/3 and $-1/3$ for u and d quarks, respectively; $\mathbf{x}_q$ is the relative separation of the electron and the corresponding quark and $\mathbf{K}_q$ is $\mathbf{P}_e + \mathbf{P}_u$ for the u quark and $\mathbf{P}_e - \mathbf{P}_d$ for the d quark ($\mathbf{P}_u$ and $\mathbf{P}_d$ are the momenta of the corresponding quarks). The expression (10) for the values of $Q = +1$ and $\mathbf{K}_q = \mathbf{K} = \mathbf{P}_e + \mathbf{P}_p$, with $\mathbf{P}_p$ the momentum of the proton, formally coincides with (28) of [6]. However, in [6] this was used for the *overall* electron–proton potential.

We should emphasize that the expressions (10) are valid up to the first order in θ. Expanding (10) to the first order in θ, it is evident that the effective Coulomb potential of the proton and electron does not only depend on the hydrogen atom center of mass momentum, which is the sum of the electron and proton momenta, $\mathbf{P}_e + \mathbf{P}_p = \mathbf{P}_e + \mathbf{P}_{u_1} + \mathbf{P}_{u_2} + \mathbf{P}_d$, invalidating the result of [6].

As a by-product, based on a similar quark model consideration on the electric dipole moment of the neutron, one obtains the non-commutativity correction

$$\mathbf{d}_n^{\rm NC} \simeq -\sum_i |Q_i| \theta \times \mathbf{P}_{\mathbf{q_i}}$$

with $|\mathbf{P}_u| \sim |\mathbf{P}_d| \sim \Lambda_{\rm QCD} \sim 200\,{\rm MeV}$ and therefore in this model $|\mathbf{d}_n^{\rm NC}| \sim e\frac{\Lambda_{\rm QCD}}{\Lambda_{\rm NC}^2}$. Using the experimental upper bound of $|\mathbf{d}_n| < 0.63 \times 10^{-25}\, e\,{\rm cm}$ [13], one obtains the lower bound $\Lambda_{\rm NC} \gtrsim 200\,{\rm TeV}$.

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[1] One may also try to use the expressions given in [10,11] for the vertices. In that case, although the numerical coefficients (for the second term in (8), as derived from (9) and (10)) would be slightly different from what we present here, the order of the bound on $\Lambda_{\rm NC}$ obtained in this way would be the same.